# Web-based Supplementary Materials for "Quantile Regression for Left-truncated Semi-Competing Risks Data" by Ruosha Li and Limin Peng

Define

$$
\begin{aligned}
\boldsymbol{S}_n(\boldsymbol{b},\tau) &= n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i^*[I\{X_i^* \le g(\boldsymbol{Z}_i^{*T}\boldsymbol{b}), \delta_i^*\eta_i^* = 1\}\widehat{G}^{-1}(Y_i^*)\widehat{\alpha}(\boldsymbol{Z}_i^*) - \tau], \\
\boldsymbol{S}_n^G(\boldsymbol{b},\tau) &= n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i^*[I\{X_i^* \le g(\boldsymbol{Z}_i^{*T}\boldsymbol{b}), \delta_i^*\eta_i^* = 1\}G^{-1}(Y_i^*)\alpha(\boldsymbol{Z}_i^*) - \tau], \\
\boldsymbol{\mu}(\boldsymbol{b},\tau) &= n^{-1/2}E\{\boldsymbol{S}_n{}^G(\boldsymbol{b},\tau)\}.
\end{aligned}
$$

Without loss of generality, we assume $a_Y = 0$. Let $\mathcal{T} = (0,\nu]$, $\mathcal{Z} = \{\boldsymbol{z} : ||\boldsymbol{z}|| \le M\}$.

## 1 Web Appendix A: Proof of Theorem 1

The first step is to sort out the asymptotic properties of $\widehat{\alpha}(\boldsymbol{z})/\widehat{G}(y)$. To this end, we need to look at each specific element of this plug-in weight. Define $R(y) = P(L < y \le Y|L < Y)$, it follows from *C2* that $\inf_{y\in(0,\nu]}R(y)$ is bounded away from 0. Define

$$
\widetilde{L}_i(y) = \int_y^{\nu}\frac{I(L_i^* < u \le Y_i^*)}{\{R(u)\}^2}F_L^*(\mathrm{d}u) - \frac{I(L_i^* > y)}{R(L_i^*)}
$$

where $F_L^*(l) = P(L \le l|L < Y)$. By the results in Chao (1987), for the Kaplan-Meier Estimators $\widehat{F}_L(\cdot)$, we have

$$
\widehat{F}_L(t) - F_L(t) = \frac{1}{n}\sum_{i=1}^{n}F_L(t)\widetilde{L}_i(t) + o_p^{\mathcal{T}}(n^{-1/2}). \qquad (A.1)
$$

Here and in the sequel, $o_p^{\mathcal{S}}(n^{-1/2})$ means root $n$ convergence to 0 in probability uniformly on set $\mathcal{S}$.

Now by (A.1) and (A.2), we can apply Taylor expansion to show

$$\begin{aligned}\widehat{\alpha}(\boldsymbol{z}) - \alpha(\boldsymbol{z}) &= \frac{1}{n}\sum_{i=1}^{n}\int_0^{\nu}\big[\xi_i(u,\boldsymbol{z})\mathrm{d}F_L(u) + S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(u)\mathrm{d}\{F_L(u)\widetilde{L}_i(u)\}\big] + o_p^{\mathcal{Z}}(n^{-1/2}) \\ &\equiv \frac{1}{n}\sum_{i=1}^{n} a_i(\boldsymbol{z}) + o_p^{\mathcal{Z}}(n^{-1/2}).\end{aligned}$$

It is not hard to see $Ea_i(\boldsymbol{z}) = 0$ from $E\xi_i(t,\boldsymbol{z}) = 0$ and $E\widetilde{L}_i(t) = 0$. Following this, we combine Taylor expansion and some algebraic manipulations to show

$$\begin{aligned}\frac{\widehat{\alpha}(\boldsymbol{z})}{\widehat{S}_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(y)} - \frac{\alpha(\boldsymbol{z})}{S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(y)} &= \frac{1}{n}\sum_{i=1}^{n}\Big\{\frac{a_i(\boldsymbol{z})}{S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(y)} - \frac{\alpha(\boldsymbol{z})\xi_i(y,\boldsymbol{z})}{S^2_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(y)}\Big\} + o_p^{\mathcal{T}\times\mathcal{Z}}(n^{-1/2}) \\ &\equiv \frac{1}{n}\sum_{i=1}^{n}\kappa_i(y,\boldsymbol{z}) + o_p^{\mathcal{T}\times\mathcal{Z}}(n^{-1/2}),\end{aligned}$$

with $E\kappa_i(y,\boldsymbol{z}) = 0$. We can further show

$$\begin{aligned}&\widehat{G}(y) - G(y) \\ =\ & \frac{1}{n}\sum_{i=1}^{n}\Big\{I(L_i^* < y \le Y_i^*)\frac{\alpha(\boldsymbol{Z}_i^*)}{S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(Y_i^*)} - G(y)\Big\} + \frac{1}{n}\sum_{j=1}^{n} I(L_j^* < y \le Y_j^*)\frac{1}{n}\sum_{i=1}^{n}\kappa_i(Y_j^*,\boldsymbol{Z}_j^*) \\ & \qquad + o_p^{\mathcal{T}}(n^{-1/2}) \\ =\ & \frac{1}{n}\sum_{i=1}^{n}\Big\{I(L_i^* < y \le Y_i^*)\frac{\alpha(\boldsymbol{Z}_i^*)}{S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(Y_i^*)} - G(y)\Big\} + \frac{1}{n}\sum_{i=1}^{n}\Big\{\frac{1}{n}\sum_{j=1}^{n}\kappa_i(Y_j^*,\boldsymbol{Z}_j^*)I(L_j^* < y \le Y_j^*)\Big\} \\ & \qquad + o_p^{\mathcal{T}}(n^{-1/2}) \\ =\ & \frac{1}{n}\sum_{i=1}^{n} g_i(y) + o_p^{\mathcal{T}}(n^{-1/2}) \qquad \text{(A.3)}\end{aligned}$$

where

$$g_i(y) \equiv \Big\{I(L_i^* < y \le Y_i^*)\frac{\alpha(\boldsymbol{Z}_i^*)}{S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(Y_i^*)} - G(y)\Big\} + E_{\widetilde{\boldsymbol{\omega}}_j^*}\big\{\kappa_i(Y_j^*,\boldsymbol{Z}_j^*)I(L_j^* < y \le Y_j^*)\big\},$$

with $\widetilde{\boldsymbol{\omega}}_i^*$ denoting $(L_i^*, X_i^*, Y_i^*, \delta_i^*, \eta_i^*, \boldsymbol{Z}_i^*)$ and $E_{\widetilde{\boldsymbol{\omega}}_j^*}$ representing the expectation over $\widetilde{\boldsymbol{\omega}}_j^*$, $j = 1, 2, ...n$. Noting that $E\big\{I(L_i^* < y \le Y_i^*)\alpha(\boldsymbol{Z}_i^*)/S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(Y_i^*)\big\} = G(y)$ and

$$E_{\widetilde{\boldsymbol{\omega}}_i^*}\big[E_{\widetilde{\boldsymbol{\omega}}_j^*}\{\kappa_i(Y_j^*,\boldsymbol{Z}_j^*)I(L_j^* < y \le Y_j^*)\}\big] = E_{\widetilde{\boldsymbol{\omega}}_j^*}\big[I(L_j^* < y \le Y_j^*)E_{\widetilde{\boldsymbol{\omega}}_i^*}\{\kappa_i(Y_j^*,\boldsymbol{Z}_j^*)\}\big] = 0,$$

we have $Eg_i(y) = 0$. We would show later that the functional class $\{g_i(y) : y \in \mathcal{T}\}$ is Donsker thus Glivenko-cantelli, therefore $\widehat{G}(y)$ is uniformly consistent for $G(y)$ on $\mathcal{T}$.

Combining the above and Taylor expansion, we have

$$\widehat{\alpha}(\boldsymbol{z})/\widehat{G}(y) - \alpha(\boldsymbol{z})/G(y) = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{a_i(\boldsymbol{z})}{G(y)} - \frac{\alpha(\boldsymbol{z})g_i(y)}{G(y)^2}\right\} + o_p^{\mathcal{T}\times\mathcal{Z}}(n^{-1/2}) \equiv \frac{1}{n}\sum_{i=1}^{n} w_i(y,\boldsymbol{z}) + o_p^{\mathcal{T}\times\mathcal{Z}}(n^{-1/2}) \quad (A.4)$$

with $Ew_i(y,\boldsymbol{z}) = 0$.

Next, we claim that $\{w_i(y,\boldsymbol{z}), \boldsymbol{z} \in \mathcal{Z}, y \in \mathcal{T}\}$ form a Donsker class (Van der Vaart and Wellner, 1996). Then, by the functional law of the iterated logarithm (Goodman et al., 1981), (A.4) implies $\sup_{\boldsymbol{z}\in\mathcal{Z}, y\in\mathcal{T}} |\widehat{\alpha}(\boldsymbol{z})/\widehat{G}(y) - \alpha(\boldsymbol{z})/G(y)| = o(n^{-1/2+r})$ for $0 < r < \frac{1}{2}$ and consequently

$$\sup_{\tau,\boldsymbol{b}} ||n^{-1/2}\boldsymbol{S}_n(\boldsymbol{b},\tau) - n^{-1/2}\boldsymbol{S}_n^G(\boldsymbol{b},\tau)|| = o(n^{-1/2+r}), a.s. \quad (A.5)$$

To show $\{w_i(y,\boldsymbol{z}), \boldsymbol{z} \in \mathcal{Z}, y \in \mathcal{T}\}$ is Donsker, we first need to prove that $\{a_i(\boldsymbol{z}) : \boldsymbol{z} \in \mathcal{Z}\}$ forms a Donsker class provided $\{\xi_i(t,\boldsymbol{z}) : t \in \mathcal{T}, \boldsymbol{z} \in \mathcal{Z}\}$ is Donsker. We first examine the component, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^{\nu}\xi_i(u,\boldsymbol{z})\mathrm{d}F_L(u)$, in $a_i(\boldsymbol{z})$ and show its weak convergence to a tight Gaussian process. It is easy to see $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i(t,\boldsymbol{z}) \rightsquigarrow \varphi(t,\boldsymbol{z})$ according to Donsker's property, where $\varphi(t,\boldsymbol{z})$ is a tight Gaussian process and $\rightsquigarrow$ means converge weakly. Note that

$$\sup_{\boldsymbol{z}\in\mathcal{Z}} |\int_0^{\nu}\{x(u,\boldsymbol{z}) - y(u,\boldsymbol{z})\}\mathrm{d}F_L(u)| \leq \sup_{t\in\mathcal{T},\boldsymbol{z}\in\mathcal{Z}} |x(t,\boldsymbol{z}) - y(t,\boldsymbol{z})|; \quad x, y \in \ell^{\infty}(\mathcal{T}\times\mathcal{Z}).$$

Then the map $\pi$ that maps $x(t,\boldsymbol{z})$ to $\int_0^{\nu} x(u,\boldsymbol{z})dF_L(u)$ is a continuous map from $\ell^{\infty}(\mathcal{T}\times\mathcal{Z})$ to $\ell^{\infty}(\mathcal{Z})$. Therefore $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^{\nu}\xi_i(u,\boldsymbol{z})dF_L(u) \rightsquigarrow \int_0^{\nu}\varphi(u,\boldsymbol{z})dF_L(u)$ according to the Continuous Mapping Theorem. Since $\pi$ is a linear map, $\int_0^{\nu}\varphi(u,\boldsymbol{z})dF_L(u)$ is a mean zero Gaussian process. The continuity of $\pi$ further ensures the asymptotic tightness of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^{\nu}\xi_i(u,\boldsymbol{z})dF_L(u)$ and also the tightness of $\int_0^{\nu}\varphi(u,\boldsymbol{z})dF_L(u)$. Similar arguments can be applied to other components of $a_i(\boldsymbol{z})$. Hence, $\{a_i(\boldsymbol{z}) : \boldsymbol{z} \in \mathcal{Z}\}$ is a Donsker class.

By similar arguments and the boundness of $S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}^{-1}(y)$ on $\mathcal{T}\times\mathcal{Z}$, we can show $\{g_i(y), y \in \mathcal{T}\}$ forms a Donsker's class. Since Donsker implies Glivenko-cantelli (Van der Vaart and Wellner, 1996) and $Eg_i(y) = 0$, $\widehat{G}(y)$ is uniformly consistent for $G(y)$ on $y \in (0,\nu]$. It follows that $\{a_i(\boldsymbol{z})/G(y) - \alpha(\boldsymbol{z})g_i(y)/G(y)^2, z \in \mathcal{Z}, y \in \mathcal{T}\}$ is also Donsker, because Donsker's property is preserved under Lipschitz transformations, and both $G(y)^{-1}$ and $\alpha(\boldsymbol{z})$ are bounded on $\mathcal{T}\times\mathcal{Z}$. Therefore, we can see (A.5) holds.

Define $\mathcal{F} = \big\{\boldsymbol{Z}_i^*[I\{X_i^* \leq g(\boldsymbol{Z}_i^{*T}b), \delta_i^*\eta_i^* = 1\}G^{-1}(Y_i^*)\alpha(\boldsymbol{Z}_i^*) - \tau],\ \boldsymbol{b} \in R^{p+1}, \tau \in [\tau_L, \tau_U]\big\}$. The function class $\mathcal{F}$ is Donsker and thus Glivenko-Cantelli because the class of indicator functions is Donsker, $\boldsymbol{Z}_i^*$, $\alpha(\boldsymbol{Z}_i^*)$ are uniformly bounded and $G(Y_i^*)$ is uniformly bounded from 0 (Van der Vaart and Wellner, 1996). By the Glivenko-Cantelli Theorem, $\sup_{\tau,\boldsymbol{b}}||n^{-1/2}\boldsymbol{S}_n^G(\boldsymbol{b},\tau) - \boldsymbol{\mu}(\boldsymbol{b},\tau)|| = o(1), a.s.$ and thus $\sup_{\tau,\boldsymbol{b}}||n^{-1/2}\boldsymbol{S}_n(\boldsymbol{b},\tau) - \boldsymbol{\mu}(\boldsymbol{b},\tau)|| = o(1), a.s.$ follows from (A.5). This, coupled with the fact that $\boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\} = 0$ and $n^{-1/2}\ \boldsymbol{S}_n\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} = o(1), a.s.$, implies that

$$\sup_{\tau \in [\tau_L, \tau_U]} \|\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}\| = o(1), \ a.s. \qquad (A.6)$$

Following the same lines of Peng and Fine (2009), we can show that Condition *C4* and the monotonicity of $\boldsymbol{\mu}(\boldsymbol{b}, \tau)$ in $b$ imply

$$\inf_{\boldsymbol{b} \notin \mathcal{B}(\rho_0), \tau \in [\tau_L, \tau_U]} ||\boldsymbol{\mu}(\boldsymbol{b}, \tau) - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)\}|| > c_0 \rho_0.$$

Consequently, $\{\widehat{\boldsymbol{\beta}}(\tau) : \tau \in [\tau_L, \tau_U]\} \subseteq \mathcal{B}(\rho_0)$ for n large enough with probability 1. Applying Taylor expansion to $\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}(\tau), \tau\}$ around $\boldsymbol{\beta}_0(\tau)$ gives

$$\begin{aligned} \sup_{\tau \in [\tau_L, \tau_U]} ||\widehat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)|| &= \sup_{\tau \in [\tau_L, \tau_U]} ||\boldsymbol{A}\{\check{\boldsymbol{\beta}}(\tau)\}^{-1}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)]|| \\ &\le c_0^{-1} \sup_{\tau \in [\tau_L, \tau_U]} ||\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau)||, \end{aligned}$$

where $\check{\boldsymbol{\beta}}(\tau)$ lies between $\widehat{\boldsymbol{\beta}}(\tau)$ and $\boldsymbol{\beta}_0(\tau)$ and is therefore within $\mathcal{B}(\rho_0)$. Note that the last inequality holds by condition *C5*. The uniform convergence of $\widehat{\boldsymbol{\beta}}(\tau)$ to $\boldsymbol{\beta}_0(\tau)$ for $\tau \in [\tau_L, \tau_U]$ then follows from (A.6).

# 2 Web Appendix B: Proof of Theorem 2

For simplicity, we write $W_j = W(Y_j^*, \boldsymbol{Z}_j^*)$, $\widehat{W}_j = \widehat{W}(Y_j^*, \boldsymbol{Z}_j^*)$ and $w_{ij} = a_i(\boldsymbol{Z}_j^*)/G(Y_j^*) - \alpha(\boldsymbol{Z}_j^*)$ $g_i(Y_j^*)/G(Y_j^*)^2$. Let $\approx$ denote asymptotic equivalence unformly in $\tau \in [\tau_L, \tau_U]$.

First, by (A.4), simple algebraic manipulations show that

$$\begin{aligned}
\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau),\tau\} &= n^{-1/2}\sum_{j=1}^{n}\boldsymbol{Z}_j^*\big(I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_j^*\eta_j^*=1]\widehat{W}_j^{-1}-\tau\big)\\
&= n^{-1/2}\sum_{j=1}^{n}\boldsymbol{Z}_j^* I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_j^*\eta_j^*=1](\widehat{W}_j^{-1}-W_j^{-1})\\
&+ n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i^*\big(I[X_i^*\le g\{\boldsymbol{Z}_i^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_i^*\eta_i^*=1]W_i^{-1}-\tau\big)\\
&\approx n^{-1/2}\sum_{j=1}^{n}\boldsymbol{Z}_j^* I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_j^*\eta_j^*=1]\sum_{i=1}^{n}w_{ij}/n\\
&+ n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Z}_i^*\big(I[X_i^*\le g\{\boldsymbol{Z}_i^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_i^*\eta_i^*=1]W_i^{-1}-\tau\big)\\
&\approx n^{-1/2}\sum_{i=1}^{n}\Big\{\boldsymbol{Z}_i^*\big(I[X_i^*\le g\{\boldsymbol{Z}_i^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_i^*\eta_i^*=1]W_i^{-1}-\tau\big)\\
&+ \frac{1}{n}\sum_{j=1}^{n}\big(\boldsymbol{Z}_j^* I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_j^*\eta_j^*=1]w_{ij}\big)\Big\}
\end{aligned}$$

An application of the Glivenko-Cantelli Theorem to $\frac{1}{n}\sum_{j=1}^{n}\big(\boldsymbol{Z}_j^* I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_j^*\eta_j^*=1]w_{ij}\big)$ gives

$$\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau),\tau\}\approx n^{-1/2}\sum_{i=1}^{n}\{\boldsymbol{\zeta}_{1i}(\tau)+\boldsymbol{\zeta}_{2i}(\tau)\},$$

where $\boldsymbol{\zeta}_{1i}(\tau)=\boldsymbol{Z}_i^*\big(I[X_i^*\le g\{\boldsymbol{Z}_i^{*T}\boldsymbol{\beta}_0(\tau)\},\delta_i^*\eta_i^*=1]W_i^{-1}-\tau\big)$ and $\boldsymbol{\zeta}_{2i}(\tau)=E_{\widetilde{\boldsymbol{\omega}}_j^*}\big(\boldsymbol{Z}_j^* I[X_j^*\le g\{\boldsymbol{Z}_j^{*T}\boldsymbol{\beta}_0(\tau)\},\ \delta_j^*\eta_j^*=1]w_{ij}\big)$ with $\widetilde{\boldsymbol{\omega}}_i^*$ denoting $\{L_i^*,X_i^*,Y_i^*,\delta_i^*,\eta_i^*,\boldsymbol{Z}_i^*\}$ and $E_{\widetilde{\boldsymbol{\omega}}_j^*}$ representing the expectation over $\widetilde{\boldsymbol{\omega}}_j^*,\ j=1,2,...n$. Following similar arguments for $a_i(\boldsymbol{z})$ in the proof of Theorem 1, we can show that $\{\boldsymbol{\zeta}_{1i}(\boldsymbol{z})+\boldsymbol{\zeta}_{2i}(\boldsymbol{z}),\ \boldsymbol{z}\in\mathcal{Z}\}$ is also a Donsker class. Therefore, $\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau),\tau\}$ converges weakly to a mean zero Gaussian Process with covariance matrix $\boldsymbol{\Sigma}(\tau',\tau)=E[\boldsymbol{\zeta}(\tau')\boldsymbol{\zeta}(\tau)]$, where $\boldsymbol{\zeta}_i(\tau)=\boldsymbol{\zeta}_{1i}(\tau)+\boldsymbol{\zeta}_{2i}(\tau)$ $(i=1,\ldots,n)$.

Next, we establish the asymptotic linearity of $\boldsymbol{S}_n^G(\boldsymbol{b},\tau)$ in the vicinity of $\boldsymbol{b}=\boldsymbol{\beta}_0(\tau)$; that is, for any positive sequence $\{d_n\}_{n=1}^n$ such that $d_n\to 0$,

$$\sup_{b,b'\in\mathcal{B}(\rho_0);||b-b'||\le d_n}||\{\boldsymbol{S}_n^G(\boldsymbol{b},\tau)-\boldsymbol{S}_n^G(b',\tau)\}-n^{1/2}\{\boldsymbol{\mu}(\boldsymbol{b},\tau)-\boldsymbol{\mu}(b',\tau)\}||=o_p(1). \qquad (A.7)$$

The proof for (A.7) greatly resembles the lines of Alexander (1984) and Lai and Ying (1988). The key is to show

$$Var\left\{\boldsymbol{Z}_i^*[I\{X_i^*\le g(\boldsymbol{Z}_i^{*T}\boldsymbol{b}),\delta_i^*\eta_i^*=1\}-I\{X_i^*\le g(\boldsymbol{Z}_i^{*T}\boldsymbol{b}'),\delta_i^*\eta_i^*=1\}]W^{-1}(Y_i^*,\boldsymbol{Z}_i^*)\right\}\le G_0||\boldsymbol{b}-\boldsymbol{b}'||.$$

This can be verified by using the uniform boundedness of the subdistribution density $f_1(t|\boldsymbol{Z})$, $\boldsymbol{Z}_i$ and $W(y, \boldsymbol{z})$.

It follows from (A.7) that

$$\begin{aligned}
& \boldsymbol{S}_n\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau), \tau\} \\
= & \; n^{-1/2} \sum_{i=1}^{n} \boldsymbol{Z}_i^* W_i^{-1} I\big(\delta_i^* \eta_i^* = 1\big)(I[X_i^* \leq g\{\boldsymbol{Z}_i^* \widehat{\boldsymbol{\beta}}(\tau)\}] - I[X_i^* \leq g\{\boldsymbol{Z}_i^* \boldsymbol{\beta}_0(\tau)\}]\big) \\
+ & \; n^{-1/2} \sum_{i=1}^{n} \boldsymbol{Z}_i^* I(\delta_i^* \eta_i^* = 1)(I[X_i^* \leq g\{\boldsymbol{Z}_i^* \widehat{\boldsymbol{\beta}}(\tau)\}] - I[X_i^* \leq g\{\boldsymbol{Z}_i^* \boldsymbol{\beta}_0(\tau)\}])(\widehat{W}_i^{-1} - W_i^{-1}) \\
\approx & \; n^{1/2}[\boldsymbol{\mu}\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau), \tau\}]
\end{aligned}$$

Along with the fact that $\boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\} = \{\partial \boldsymbol{\mu}(\boldsymbol{b}, \tau)/\partial \boldsymbol{b}\}|_{\boldsymbol{b}=\boldsymbol{\beta}_0(\tau)}$ and $\widehat{\boldsymbol{\beta}}(\tau)$ uniformly converges to $\boldsymbol{\beta}_0(\tau)$, a Taylor expansion of $\boldsymbol{\mu}(\boldsymbol{b}, \tau)$ around $\boldsymbol{b} = \boldsymbol{\beta}_0(\tau)$ gives that

$$\boldsymbol{S}_n\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} - \boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau), \tau\} \approx \boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\} n^{1/2}\{\widehat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}.$$

This implies

$$n^{1/2}\{\widehat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\} \approx -\boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\}^{-1} \boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau), \tau\}, \qquad (A.8)$$

and then $n^{1/2}\{\widehat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}$ converges weakly to a mean zero Gaussian process with covariance matrix

$$\boldsymbol{A}\{\boldsymbol{\beta}_0(\tau')\}^{-1} \boldsymbol{\Sigma}(\tau', \tau) \boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\}^{-T}.$$

# 3 Web Appendix C: Justification for the Proposed Covariance Matrix Estimate

With the strict convexity condition in *C4*, it is implied from the proof of Theorem 1 that $\big\{\boldsymbol{S}_n^{-1}\{\boldsymbol{e}_{n,j}(\tau), \tau\}, \tau \in [\tau_L, \tau_U]\big\}$ falls in $\mathcal{B}(\rho_0)$ with probability 1 when n is large enough, and uniformly converges to $\boldsymbol{\beta}_0(\tau)$, $j = 1, 2, ..p + 1$. Denote $\boldsymbol{b}_{n,j}(\tau) = \boldsymbol{S}_n^{-1}\{\boldsymbol{e}_{n,j}(\tau), \tau\}$. Using arguments similar to those for (A.7), we can show that

$$\boldsymbol{S}_n\{\boldsymbol{b}_{n,j}(\tau), \tau\} - \boldsymbol{S}_n\{\widehat{\boldsymbol{\beta}}(\tau), \tau\} \approx \boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\} n^{1/2} \left[\boldsymbol{b}_{n,j}(\tau) - \widehat{\boldsymbol{\beta}}(\tau)\right].$$

By the definition of $\boldsymbol{D}_n(\tau)$ and $\boldsymbol{E}_n(\tau)$, this implies $\boldsymbol{E}_n(\tau) \approx \sqrt{n}\boldsymbol{A}\{\boldsymbol{\beta}_0(\tau)\}\boldsymbol{D}_n(\tau)$. Thus $\sqrt{n}\boldsymbol{D}_n(\tau)\boldsymbol{E}_n^{-1}(\tau)$ is a consistent estimator for $A^{-1}\{\boldsymbol{\beta}_0(\tau)\}$. It follows immediately that

$$n\boldsymbol{D}_n(\tau')\boldsymbol{E}_n^{-1}(\tau')\widehat{\boldsymbol{\Sigma}}(\tau', \tau)\boldsymbol{E}_n(\tau)^{-1}\boldsymbol{D}_n(\tau)^T$$

is a consistent estimator for the asymptotic covariance matrix of $n^{1/2}\{\widehat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}$.

# 4 Web Appendix D: The Form of $\xi_i(t, \boldsymbol{z})$ under the Cox Proportional Hazard Model

Here we present the form of $\xi_i(t, \boldsymbol{z})$ when the Cox proportional hazards model is assumed for $T_2$. In that case, $P(T_2 > t|\boldsymbol{Z}) = \exp\{-\Lambda_0(t)\exp(\boldsymbol{\gamma}_0^T \widetilde{\boldsymbol{Z}})\}$, where $\boldsymbol{\gamma}_0$ is the $p \times 1$ regression coefficient, and $\Lambda_0(t)$ is the baseline cumulative hazard function.

Define $N_i(t) = I(Y_i^* \le t)\eta_i^*$, $M_i(t) = N_i(t) - \Lambda_0(t)\exp(\boldsymbol{\gamma}_0^T \widetilde{\boldsymbol{Z}}_i^*)$, $R_i(t) = I(L_i^* < t \le Y_i^*)$, $\boldsymbol{S}^{(j)}(\boldsymbol{\gamma}, t) = 1/n \sum_{i=1}^n (\widetilde{\boldsymbol{Z}}_i^*)^{\otimes j} \times R_i(t) \exp\{\boldsymbol{\gamma}^T \widetilde{\boldsymbol{Z}}_i^*\}$, where $j = 0, 1, 2$. Let $\boldsymbol{E}(\boldsymbol{\gamma}, t) = \boldsymbol{S}^{(1)}(\boldsymbol{\gamma}, t)/S^{(0)}(\boldsymbol{\gamma}, t)$, $\boldsymbol{V}(\boldsymbol{\gamma}, t) = \dfrac{\boldsymbol{S}^{(2)}(\boldsymbol{\gamma}, t)}{S^{(0)}(\boldsymbol{\gamma}, t)} - \boldsymbol{E}(\boldsymbol{\gamma}, t)^{\otimes 2}$, $\boldsymbol{G} = \int_0^\nu \boldsymbol{V}(\boldsymbol{\gamma}_0, t) S^{(0)}(\boldsymbol{\gamma}_0, t) \mathrm{d}\Lambda_0(t)$, and $\boldsymbol{P}(t) = -\int_0^t \boldsymbol{E}(\boldsymbol{\gamma}_0, u)\mathrm{d}\Lambda_0(u)$. Let $\boldsymbol{\eta}_0(t) = \{\Lambda_0(t), \boldsymbol{\gamma}_0\}^T$. Adapting Andersen and Gill (1982)'s results, we can show

$$
\begin{aligned}
& \widehat{\boldsymbol{\eta}}(t) - \boldsymbol{\eta}(t) \\
= & \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n [\boldsymbol{P}(t)^T \int_0^\nu \boldsymbol{G}^{-1}\{\widetilde{\boldsymbol{Z}}_i^* - \boldsymbol{E}(\boldsymbol{\gamma}_0, u)\}\mathrm{d}M_i(u) + \int_0^t \dfrac{1}{S^{(0)}(\boldsymbol{\gamma}_0, u)}\mathrm{d}M_i(u)] \\ \frac{1}{n}\sum_{i=1}^n \int_0^\nu \boldsymbol{G}^{-1}\{\widetilde{\boldsymbol{Z}}_i^* - \boldsymbol{E}(\boldsymbol{\gamma}_0, u)\}\mathrm{d}M_i(u)\} \end{pmatrix} + o_p^{\mathcal{T}}(n^{-1/2}) \\
\equiv & \frac{1}{n}\sum_{i=1}^n \{p_i^{(1)}(t), \boldsymbol{p}_i^{(2)}\}^T + o_p^{\mathcal{T}}(n^{-1/2}), \qquad (A.3)
\end{aligned}
$$

and $\left\{\left(p_i^{(1)}(t), \boldsymbol{p}_i^{(2)}\right)^T, t \in \mathcal{T}\right\}$ forms a Donsker's class.

Applying Taylor expansions gives that

$$
\widehat{S}_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(t) - S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(t) = \frac{1}{n}\sum_{i=1}^n [-\exp(\boldsymbol{\gamma}_0{}^T z) S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(t)\{p_i^{(1)}(t) + \Lambda_0(t)\boldsymbol{p}_i^{(2)^T}\boldsymbol{z}\}] + o_p^{\mathcal{T}\times\mathcal{Z}}(n^{-1/2}).
$$

Therefore, it is easy to see that the influence function of $\widehat{S}_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(t)$ is given by

$$
\xi_i(t, \boldsymbol{z}) \equiv -\exp(\boldsymbol{\gamma}_0{}^T \boldsymbol{z}) S_{T_2|\boldsymbol{Z}=\boldsymbol{z}}(t)\{p_i^{(1)}(t) + \Lambda_0(t)\boldsymbol{p}_i^{(2)^T}\boldsymbol{z}\}.
$$

We can show $\{\xi_i(t, \boldsymbol{z}), t \in \mathcal{T}, \boldsymbol{z} \in \mathcal{Z}\}$ is Donsker following similar arguments for $a_i(\boldsymbol{z})$ in Web Appendix A.

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