Web based appendix

Applications and Extensions of Chao's Moment Estimator for the Size of a Closed Population

Web Appendix A: Formulas used to calculate the densities of Figure 1

For Darroch's model the mixing density is given by

$$\begin{aligned} f(x) &= \frac{\sum \binom{t}{i} \exp(\beta i + \tau i^2/2) \exp(i(x-\beta) - (\tau i^2)/2) \exp\{-(x-\beta)^2/(2\tau)\}/\sqrt{2\pi\tau}}{\sum \binom{t}{i} \exp(\beta i + \tau i^2/2)} \\ &= \frac{\sum \binom{t}{i} \exp\{ix - (x-\beta)^2/(2\tau)\}/\sqrt{2\pi\tau}}{\sum \binom{t}{i} \exp(\beta i + \tau i^2/2)} \end{aligned}$$

For the Poisson model it is $\beta + X$ where the probability function of X is

$$\begin{aligned} \Pr(\mathrm{X} = (\log 2)\mathrm{k}) &= \frac{\sum \binom{t}{i} \exp(\beta i + \tau(2^i - 1)) 2^{ki} \exp(-\tau(2^i-1))\tau^k \exp(-\tau)/k!}{\sum \binom{t}{i} \exp(\beta i + \tau(2^i-1))} \\ &= \frac{\sum \binom{t}{i} \exp(\beta i)(2^i\tau)^k/k!}{\sum \binom{t}{i} \exp(\beta i + \tau 2^i)} \end{aligned}$$

for $k = 0, 1, \ldots$ and for the gamma model

$$\begin{aligned} f(x) &= \frac{\sum \binom{t}{i} \exp(\beta i - \tau \log(3.5+i)) \exp(i(x-\beta) + \tau\log(3.5+i))(\beta - x)^\tau \exp\{3.5(x-\beta)\}/\Gamma(\tau)}{\sum \binom{t}{i} \exp(\beta i - \tau\log(3.5+i))} \\ &= \frac{\sum \binom{t}{i} \exp(ix)(\beta-x)^\tau \exp\{3.5(x-\beta)\}/\Gamma(\tau)}{\sum \binom{t}{i} \exp(\beta i - \tau\log(3.5+i))}, \end{aligned}$$

for $x < \beta$.

Web Appendix B: Proof of Proposition 1.

The derivations rely on the following following that generalizes results reported in Seber and Manly (1985) and in Rivest et al. (1995).

Lemma *Let (X_1, X_2, X_3, X_4) be random variables with a multinomial distribution with parameters N and (p_1, p_2, p_3, p_4), with $\sum p_i = 1$; if t_1, t_2, and t_3 are non negative integers then*

$$\begin{aligned} &E\left(\frac{X_1(X_1-1)\ldots(X_1-t_1+1)X_2(X_2-1)\ldots(X_2-t_2+1)}{(X_3+1)\ldots(X_1+t_3)}\right) \\ &= \frac{p_1^{t_1}p_2^{t_2}}{p_3^{t_3}} \frac{N!}{(N-t_1-t_2+t_3)!}\left(1 - \sum_{k=0}^{t_3-1} \frac{(N-t_1-t_2+t_3)!}{k!(N-t_1-t_2+t_3-k)!} p_3^k (1-p_3)^{N-t_1-t_2+t_3-k}\right). \end{aligned}$$

The strategy to construct bias corrected variance estimators is to write $\hat{N}_C = n\hat{c}$, where $\hat{c}$ is the correction factor for missed animals. Since $\hat{N}_C$ has a small bias, one can assume that the bias of $\hat{c}$ as an estimator of $c = 1 + (t-1)E(f_1)^2/\{2tE(n)E(f_2)\}$ is small. Observe that n has a binomial distribution with parameters N and $1/c$ thus $\text{Var}(n) = N(c-1)/c^2$. Now, conditioning on n gives $\text{Var}(n\hat{c}) = \text{Var}\{E(n\hat{c}|n)\} + E\{\text{Var}(n\hat{c}|n)\}$ and since the bias of $\hat{c}$ is negligible, $\text{Var}(n\hat{c}) = c^2\text{Var}(n) + E\{n^2\text{Var}(\hat{c}|n)\}$. A bias corrected variance estimate is given by

$$\begin{aligned} v(\hat{N}) &= n(\hat{c}^2 - v(\hat{c})) - n\hat{c} + n^2 v(\hat{c}) \\ &= n\hat{c}(\hat{c}-1) + n(n-1)v(\hat{c}), \end{aligned}$$

where $v(\hat{c}) = (\hat{c}-1)^2 - \hat{c}_2$ is a bias corrected estimator for the variance of $\hat{c}$ given n and

$$\hat{c}_2 = \frac{(t-1)^2 f_1(f_1-1)(f_1-2)(f_1-3)}{4t^2 n(n-1)(f_2+1)(f_2+2)}$$

is a bias corrected estimator of $(c-1)^2$ constructed by applying the lemma to the conditional distribution of f_1, f_2 given n. Thus $v(\hat{N}) = n(\hat{c}-1) + n^2(\hat{c}-1)^2 - n(n-1)\hat{c}_2$ as stated in Proposition 1.

Web Table 1: Demographic parameter estimates for the vole data

	Robust Design	Jolly-Seber Model
ϕ_1	0.82 (0.055)	0.82 (0.057)
ϕ_2	0.57 (0.068)	0.56 (0.065)
ϕ_3	0.72 (0.077)	0.70 (0.073)
ϕ_4	0.55 (0.068)	0.58 (0.068)
ϕ_5	0.99 (0.08)	-
N_1	63 (4.6)	-
N_2	69 (1.7)	75 (2.6)
N_3	60 (3.9)	60 (3.8)
N_4	67 (4.3)	63 (3.3)
N_5	54 (2.0)	56 (3.2)
N_6	92 (7.4)	-

Demographic parameter estimates, and their standard errors in parenthesis, obtained with the robust design M_h^t(Poisson) with $a = 1.5$ and with the Jolly-Seber model.

References

Rivest, L.-P., Potvin, F., Crépeau, H., and Daigle, G. (1995). Statistical methods for aerial surveys using the double-count technique to correct visibility bias. *Biometrics*, 51, 461–470

Seber, G. A. F. and Manly, B. F. J. (1985). Approximately unbiased variance estimator for the Jolly-Seber mark-recapture model: Population Size. *Statistics in Ornithology, B.J.T. Morgan and P.M. North Editors*, 363–371