Web-based Supplementary Materials for

Analysis of Failure Time Data with Multi-Level Clustering, with Application to The Child Vitamin A Intervention Trial in Nepal

by

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1. Derivation of B_{2I}

The score function for $l_2(\theta_{20})$ is

$$\begin{aligned}\frac{\partial l_2(\theta_{20})}{\partial \theta_{20}} &= I^{-1}\sum_{i=1}^{I}\sum_{j=1}^{n_i}\{\sum_{l=1}^{\delta_{ij+}}\frac{l-1}{1+(l-1)\theta_{20}}]\} + \{\sum_{k=1}^{m_{ij}}\delta_{ijk}H_{ijk}\} + \theta_{20}^{-2}\log\{R_{ij}(\theta_{20})\} \\ &\quad -(\theta_{20}^{-1}+\delta_{ij+})U_{ij}(\theta_{20})R_{ij}^{-1}(\theta_{20}),\end{aligned}$$

where $U_{ij}(\theta_{20}) = \sum_{k=1}^{m_{ij}} H_{ijk}e^{\theta_{20}H_{ijk}}$.

The minus 2nd derivative of $l_2(\theta_{20})$, denoted by $B_{2I}(\theta_{20})$, is

$$\begin{aligned}B_{2I}(\theta_{20}) &= -\frac{\partial^2 l_2(\theta_{20})}{\partial \theta_{20}^2} \\ &= I^{-1}\sum_{i=1}^{I}\sum_{j=1}^{n_i}[\sum_{l=1}^{\delta_{ij+}}\frac{(l-1)^2}{\{1+(l-1)\theta_{20}\}^2}] + 2\theta_{20}^{-3}\log\{R_{ij}(\theta_{20})\} - 2\theta_{20}^{-2}U_{ij}(\theta_{20})R_{ij}^{-1}(\theta_{20}) \\ &\quad +\{\theta_{20}^{-1}+\delta_{ij+}\}\{V_{ij}(\theta_{20})R_{ij}^{-1}(\theta_{20}) - U_{ij}^2(\theta_{20})R_{ij}^{-2}(\theta_{20})\},\end{aligned}$$

where $V_{ij}(\theta_{20}) = \sum_{k=1}^{m_{ij}} H_{ijk}^2 e^{\theta_{20}H_{ijk}}$.

The information for $l_2(\theta_{20})$, $B_2(\theta) = \lim_{I\to\infty} B_{2I}(\theta_{20})$ can be consistently estimated by $-\partial^2 \hat{l}_2(\theta_{20})/\partial\theta_{20}^2 \mid_{\theta_{20}=\hat{\theta}_2}$.

2. Derivation of $\sigma^2_{\Phi_2}$

The variance $\sigma^2_{\Phi_2}$ accounts for the variability in the estimates $\hat{\boldsymbol{\beta}}$ and $\Lambda_0(.)$, and adjusts for the correlation among failure times in each village. Following the derivation of Glidden (2000), $\sigma^2_{\Phi_2}$ is the variance of I i.i.d. random variables Φ_{2i} which have the form

$$\Phi_{2i} = \phi_{2i} + \int_0^{\nu} \pi_2(s)\, d\Psi_i(s) + \mathbf{F}_2^T \mathbf{A}^{-1} \mathbf{w}_{i++},$$

where

$$\phi_{2i} = \sum_{j=1}^{n_i} \{ \sum_{l=1}^{\delta_{ij+}} \frac{l-1}{1+(l-1)\theta_{20}} \} + \{ \sum_{k=1}^{m_{ij}} \delta_{ijk} H_{ijk} \} + \theta_{20}^{-2} \log\{R_{ij}(\theta_{20})\} - (\theta_{20}^{-1} + \delta_{ij+}) U_{ij}(\theta_{20}) R_{ij}^{-1}(\theta_{20}),$$

$\pi_2(t)$ is the pointwise limit of $\pi_{2I}(t)$ given by

$$\begin{aligned} \pi_2(t) &= \lim_{I\to\infty} \pi_{2I}(t) \\ &= I^{-1} \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} e^{\boldsymbol{\beta}^T \mathbf{z}_{ijk}} Y_{ijk}(t) \left\{ \theta_{20}^{-1} R_{ij}^{-1}(\theta_{20}) e^{\theta_{20} H_{ijk}} - (\theta_{20}^{-1} + \delta_{ij+})(1 + \theta_{20} H_{ijk}) \right. \\ &\quad \left. \times R_{ij}^{-1}(\theta_{20}) e^{\theta_{20} H_{ijk}} + (1 + \theta_{20}\delta_{ij+}) U_{ij}(\theta_{20}) R_{ij}^{-2}(\theta_{20}) e^{\theta_{20} H_{ijk}} + \delta_{ijk} \right\}, \end{aligned}$$

$\mathbf{F}_2$ is the limit of $\mathbf{F}_{2I}$ given by

$$\begin{aligned} \mathbf{F}_2 &= \lim_{I\to\infty} \mathbf{F}_{2I} \\ &= I^{-1} \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} H_{ijk} \mathbf{Z}_{ijk} \left\{ \theta_{20}^{-1} R_{ij}^{-1}(\theta_{20}) e^{\theta_{20} H_{ijk}} - (\theta_{20}^{-1} + \delta_{ij+})(1 + \theta_{20} H_{ijk}) R_{ij}^{-1}(\theta_{20}) e^{\theta_{20} H_{ijk}} \right. \\ &\quad \left. + (1 + \theta_{20}\delta_{ij+}) U_{ij}(\theta_{20}) R_{ij}^{-2}(\theta_{20}) e^{\theta_{20} H_{ijk}} + \delta_{ijk} \right\}, \end{aligned}$$

and

$$\Psi_i(t) = \int_0^t \frac{dM_{i++}(u)}{S^{(0)}(\boldsymbol{\beta})} + \mathbf{h}^T(t) \mathbf{A}^{-1} \mathbf{w}_{i++},$$

where

$$M_{i++}(t) = \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} M_{ijk}(t),$$

$$M_{ijk}(t) = N_{ijk}(t) - \int_0^t Y_{ijk}(u) e^{\boldsymbol{\beta}^T \mathbf{z}_{ijk}}\, d\Lambda_0(u),$$

$$\mathbf{h}(t) = \int_0^t \mathbf{E}(\boldsymbol{\beta}, u)\, d\Lambda_0(u),$$

$$\begin{aligned}\mathbf{A} &= \lim_{I\to\infty} \mathbf{A}_I \\ &= I^{-1}\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}\int_0^{\nu}\left\{\frac{\mathbf{S}^{(2)}(\boldsymbol{\beta},u)}{S^{(0)}(\boldsymbol{\beta},u)} - \mathbf{E}(\boldsymbol{\beta},u)^{\otimes 2}\right\} dN_{ijk}(u),\end{aligned}$$

$$\mathbf{w}_{i++} = \sum_{j=1}^{n_i}\sum_{k=1}^{m_{ij}}\int_0^{\nu}\{\mathbf{Z}_{ijk} - \mathbf{E}(\boldsymbol{\beta},u)\}\, dM_{ijk}(u).$$

3. Derivation of $\hat{\sigma}_1^{2(q)}$

First,the pseudo likelihood for θ_1 based on qth resampled dataset is given by

$$l_1^{(q)}(\theta_1) = I^{-1}\sum_{i=1}^{I}[\sum_{l=1}^{\delta_{i+}^{(q)}}\log\{1+(l-1)\theta_1\}] + \{\sum_{j=1}^{n_i}\theta_1\delta_{ijK_{ij}^{(q)}}H_{ijK_{ij}^{(q)}}\} - (\theta_1^{-1}+\delta_{i+}^{(q)})\log\{R_i^{(q)}(\theta_1)\},$$

where $K_{ij}^{(q)}, j = 1,\cdots,n_i, i = 1,\cdots,I$ are random variables taking on values $\{1,\cdots,m_{ij}\}$ with equal probability, and are used to index subjects randomly selected, with replacement, from jth household in ith village for qth resampled data set, $\delta_{i+}^{(q)} = \sum_{j=1}^{n_i}\delta_{ijK_{ij}^{(q)}}$, and $R_i^{(q)}(\theta_1) = \sum_{j=1}^{n_i}\exp\{\theta_1 H_{ijK_{ij}^{(q)}}\} - n_i + 1$. The estimate $\hat{\theta}_1^{(q)}$ of θ_{10} is obtained by maximizing $\hat{l}_1^{(q)}(\theta_1)$ where, like l_2, ˆ over l_1 indicates that $\hat{\Lambda}_0$ and $\hat{\boldsymbol{\beta}}_0$ are inserted in the likelihood.

The estimated variance $\hat{\sigma}_1^{2(q)}$ obtained from the qth resampled data set has the form

$$\hat{\sigma}_1^{2(q)} = \hat{B}_{1I}^{-1(q)}(\hat{\theta}_1^{(q)})\hat{\sigma}_{\Phi_1}^{2(q)}\hat{B}_{1I}^{-1}(\hat{\theta}_1^{(q)}).$$

The term $\hat{B}_{1I}^{-1(q)}(\hat{\theta}_1^{(q)})$ is given by

$$\begin{aligned}\hat{B}_{1I}^{(q)}(\hat{\theta}_1^{(q)}) &= -\frac{\partial^2 \hat{l}_1^{(q)}(\theta_{10})}{\partial\theta_{10}^2} \mid \theta_{10} = \hat{\theta}_1^{(q)} \\ &= I^{-1}\sum_{i=1}^{I}[\sum_{l=1}^{\delta_{i+}^{(q)}}\frac{(l-1)^2}{\{1+(l-1)\hat{\theta}_1^{(q)}\}^2}] + [\sum_{j=1}^{n_i}\frac{2\log\{\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)})\}}{\hat{\theta}_1^{(q)3}}] - \frac{2\hat{U}_i^{(q)}(\hat{\theta}_1^{(q)})}{\hat{\theta}_1^{(q)2}\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)})} \\ &\quad + \frac{\{1+\hat{\theta}_1^{(q)}\delta_{i+}^{(q)}\}\{\hat{V}_i^{(q)}(\hat{\theta}_1^{(q)})\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) - \hat{U}_i^{(q)2}(\hat{\theta}_1^{(q)})\}}{\hat{\theta}_1^{(q)}\hat{R}_i^{(q)2}(\hat{\theta}_1^{(q)})},\end{aligned}$$

where $\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) = \sum_{j=1}^{n_i} \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\}$, $\hat{U}_i^{(q)}(\hat{\theta}_1^{(q)}) = \sum_{j=1}^{n_i} \hat{H}_{ijK_{ij}^{(q)}} \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\}$, $\hat{V}_i^{(q)}(\hat{\theta}_1^{(q)}) = \sum_{j=1}^{n_i} \hat{H}^2_{ijK_{ij}^{(q)}} \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\}$, and $\hat{H}_{ijk} = e^{\hat{\boldsymbol{\beta}}^T \mathbf{Z}_{ijk}} \hat{\Lambda}_0(X_{ijk})$.

The other term $\hat{\sigma}^{2(q)}_{\Phi_1}$ has the form $\hat{\sigma}^{2(q)}_{\Phi_1} = I^{-1} \sum_{i=1}^{I} \hat{\Phi}_{1i}^{2(q)}$. The terms $\hat{\Phi}_{1i}^{(q)}, i = 1, \cdots, I$ have the expression

$$\hat{\Phi}_{1i}^{(q)} = \hat{\phi}_{1i}^{(q)} + \int_0^{\nu} \hat{\pi}_1^{(q)}(s)\, d\hat{\Psi}_i(s) + \hat{\mathbf{F}}_1^{(q)T} \hat{\mathbf{A}}^{-1} \hat{\mathbf{w}}_{i++},$$

where

$$\hat{\phi}_{1i}^{(q)} = \sum_{l=1}^{\delta_{i+}^{(q)}} \frac{(l-1)}{\{1 + (l-1)\hat{\theta}_1^{(q)}\}} + \sum_{j=1}^{n_i} \delta_{ijK_{ij}^{(q)}} \hat{H}_{ijK_{ij}^{(q)}} + \frac{\log\{\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)})\}}{\theta_1^{(q)2}} - \frac{(1 + \delta_{i+}^{(q)})\hat{U}_i(\hat{\theta}_1^{(q)})}{\hat{\theta}_1^{(q)} \hat{R}_i^{(q)}(\hat{\theta}_1^{(q)})},$$

$$\begin{aligned}\hat{\pi}_1^{(q)}(t) = I^{-1} \sum_{i=1}^{I} \sum_{j=1}^{n_i} \frac{\exp\{\hat{\boldsymbol{\beta}}^T \mathbf{Z}_{ijK_{ij}^{(q)}}\} Y_{ijK_{ij}^{(q)}}(t)}{\hat{\theta}_1^{(q)} \hat{R}_i^{(q)2}(\hat{\theta}_1^{(q)})} \Big[\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} \\ -\{1 + \hat{\theta}_1^{(q)} \delta_{i+}^{(q)}\}\{1 + \hat{\theta}_1^{(q)} H_{ijK_{ij}^{(q)}}\} \hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} \\ +\hat{\theta}_1^{(q)}\{1 + \hat{\theta}_1^{(q)} \delta_{i+}^{(q)}\} U_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} + \delta_{ijK_{ij}^{(q)}} \hat{\theta}_1^{(q)} \hat{R}_i^{(q)2}(\hat{\theta}_1^{(q)}) \Big],\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{F}}_1^{(q)} = I^{-1} \sum_{i=1}^{I} \sum_{j=1}^{n_i} \frac{\hat{H}_{ijK_{ij}^{(q)}} \mathbf{Z}_{ijK_{ij}^{(q)}}}{\hat{\theta}_1^{(q)} \hat{R}_i^{(q)2}(\hat{\theta}_1^{(q)})} \Big[\hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} \\ -\{1 + \hat{\theta}_1^{(q)} \delta_{i+}^{(q)}\}\{1 + \hat{\theta}_1^{(q)} H_{ijK_{ij}^{(q)}}\} \hat{R}_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} \\ +\hat{\theta}_1^{(q)}\{1 + \hat{\theta}_1^{(q)} \delta_{i+}^{(q)}\} U_i^{(q)}(\hat{\theta}_1^{(q)}) \exp\{\hat{\theta}_1^{(q)} \hat{H}_{ijK_{ij}^{(q)}}\} + \delta_{ijK_{ij}^{(q)}} \hat{\theta}_1^{(q)} \hat{R}_i^{(q)2}(\hat{\theta}_1^{(q)}) \Big],\end{aligned}$$

and $\hat{\Psi}_i(t), \hat{\mathbf{A}}, \hat{\mathbf{w}}_{i++}$ are the values of $\Psi_i(t), \mathbf{A}, \mathbf{w}_{i++}$ respectively, evaluated at $\boldsymbol{\beta}_0 = \hat{\boldsymbol{\beta}}$ and $\Lambda_0(t) = \hat{\Lambda}_0(t), 0 < t \le \nu$.

4. Derivation of $\Upsilon_i(t)$

By Taylor expansion of $\hat{W}(t)$ respect to $\boldsymbol{\beta}$ and θ_2 around $\boldsymbol{\beta}_0$ and θ_{20} and functional Taylor expansion of $\hat{W}(.)$ with respect to $\Lambda_0(.)$, $\hat{W}(t)$ can be approximated by

$$\begin{aligned}
\hat{W}(t) &= W(t) + \sqrt{I}\tilde{f}(t)(\hat{\theta}_2 - \theta_{20}) + \sqrt{I}\int_0^t \tilde{g}(u;t)\, d\{\hat{\Lambda}_0(u) - \Lambda_0(u)\} + \sqrt{I}\tilde{\mathbf{q}}^T(t)(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + o_p(1) \\
&= I^{-1/2}\left\{\sum_{i=1}^{I} \epsilon_i(t) + B_2(\theta_{20})^{-1}\Phi_{2i} f(t) + \int_0^t g(u;t)\, d\Psi_i(u) + \mathbf{q}^T(t)\mathbf{A}^{-1}\mathbf{w}_{i++}\right\} + o_p(1) \\
&= I^{-1/2}\sum_{i=1}^{I} \Upsilon_i(t) + o_p(1),
\end{aligned}$$

where $\epsilon_i(t) = \sum_{j=1}^{m_{ij}}\{\bar{\gamma}_{ij}(b;\boldsymbol{\beta}_0,\theta_{20}) - 1\}$, $f(.)$ is the limit of $\tilde{f}(.)$ given by

$$f(t) = \lim_{I\to\infty}\tilde{f}(t) = I^{-1}\sum_i\sum_j R_{ij}^{-1}(\theta_{20};t))\{N_{ij}(t) - \bar{\gamma}_{ij}(t;\boldsymbol{\beta}_0,\theta_{20})U_{ij}(\theta_{20};t)\},$$

where $U_{ij}(\theta_{20};t) = \sum_{k=1}^{m_{ij}} H_{ijk}(t \wedge X_{ijk})e^{\theta_{20}H_{ijk}(t\wedge X_{ijk})}$,

$g(.)$ is the limit of $\tilde{g}(.)$ given by

$$g(t) = \lim_{I\to\infty}\tilde{g}(t) = I^{-1}\sum_i\sum_j \theta_{20}R_{ij}^{-1}(\theta_{20};t))\bar{\gamma}_{ij}(t;\boldsymbol{\beta}_0,\theta_{20})\sum_{k=1}^{m_{ij}} e^{\theta_{20}H_{ijk}(t\wedge X_{ijk})+\boldsymbol{\beta}_0'\mathbf{Z}_{ijk}}Y_{ijk}(u),$$

and $\mathbf{q}(.)$ is the limit of $\tilde{\mathbf{q}}(.)$ given by

$$\mathbf{q}(t) = \lim_{I\to\infty}\tilde{q}(t) = I^{-1}\sum_i\sum_j \theta_{20}R_{ij}^{-1}(\theta_{20};t))\bar{\gamma}_{ij}(t;\boldsymbol{\beta}_0,\theta_{20})\sum_{k=1}^{m_{ij}} e^{\theta_{20}H_{ijk}(t\wedge X_{ijk})}H_{ijk}(t\wedge X_{ijk})\mathbf{Z}_{ijk}.$$

5. Simulating from power variance function distribution

Let X follow a positive stable distribution denoted by $P(\alpha,\alpha,0)$ with Laplace transform $\phi(s) = \exp(-s^\alpha)$, $0 < \alpha \le 1$. The distribution $P(\alpha,\delta,0)$ for $\delta > 0$ is defined as that of $(\delta/\alpha)^{1/\alpha}X$ which has Laplace transform $\phi(s) = \exp(-\delta s^\alpha/\alpha)$. It follows from the Laplace transform that

$$f\{x(\alpha/\delta)^{1/\alpha}\}(\alpha/\delta)^{1/\alpha}\exp(-\theta x)\exp(\delta\theta^\alpha/\alpha), \theta \ge 0, \delta > 0, 0 < \alpha \le 1$$

is a probability density, where f is the probability density function of X. The distribution is denoted by $P(\alpha,\delta,\theta)$ and has the Laplace transform $\phi(s) = \exp[-\delta\{(\theta+s)^\alpha - \theta^\alpha\}/\alpha]$.

Since the power variance function distribution is related to the positive stable distribution, it is useful to comment on how to simulate from a positive stable distribution $P(\alpha, \alpha, 0)$. Stephenson (2003) provides the following algorithm for simulating from $P(\alpha, \alpha, 0)$. Let U be a uniformly distributed over the interval $(0, \pi)$ and let W be standard exponential, independent of U. Then

$$\left[\frac{\sin\{(1-\alpha)U\}}{W}\right]^{(1-\alpha)/\alpha} \frac{(\sin \alpha U)}{(\sin U)^{1/\alpha}} \sim P(\alpha, \alpha, 0).$$

Then the algorithm to generate random deviate Z from $P(\alpha, \delta, \theta)$ is as follows:

1. Generate X from $P(\alpha, \alpha, 0)$ using the above algorithm and set $Y = (\delta/\alpha)^{1/\alpha} X$.

2. Accept $Z = Y$ with probability $e^{-\theta Y}$

Reference

Stephenson, A. (2003). Simulating multivariate extreme value distributions of logistic type. *Extremes* **6**, 49-59.